Ph.D. selection test

Department of Physics

Indian Institute of Technology, Kanpur

December 5, 2017 Time : 9:30 – 11:30 AM Maximum marks : 70

Question 1

(**A**) Consider a particle in an infinite potential well [the potential $V(x) = 0$ for $0 < x < L$, otherwise $V(x) = \infty$]. The quantum system is described by the energy eigenvalues E_n and the corresponding normalized eigenstates $\phi_n(x)$ with n = 1, 2, 3,…..

At time $t = 0$, a particle in the infinite well is in the state given by

$$\psi(x,0) = \sqrt{\frac{1}{3}}\phi_1(x) + \sqrt{\frac{1}{6}}\phi_2(x) + \sqrt{\frac{1}{2}}\phi_3(x)\ .$$

(a) Write down the expression for $\psi(x,t)$ [**1 mark**]

(b) Calculate the expectation value of the energy for the particle described by $\psi(x,t)$. Write your answer in terms of E_1. [**3 marks**]

(**B**) Consider a spherically symmetric rigid rotor with moment of inertia $I_x = I_y = I_z = I_o$. Its Hamiltonian is given by

$$H = \frac{\boldsymbol{L}^2}{2I_o}$$

with $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ is the orbital angular momentum operator.

(a) What are the energy eigenstates and eigenvalues for this quantum rigid rotor? [**1 mark**]

(b) Now suppose the moment of inertia in the z-direction becomes $I_z = (1 + \varepsilon)\, I_o$, where ($\varepsilon << 1$) and with the other two moments unchanged i.e $I_x = I_y = I_o$. What are the new energy eigenstates and eigenvalues? [**5 marks**]

Question 2

A neutral spherical ball with radius R and dielectric permittivity ε_2 is kept inside an infinite dielectric media with permittivity ε_1 . The whole system is placed in an electric field which is uniform far away from the sphere and is given by $\vec{E} = E_0\,\hat{z}$. After solving the Laplace's equation in spherical coordinates, the following solutions are obtained for the potential:

$$V(r \le R) = -\frac{3\varepsilon_1}{\varepsilon_2 + 2\varepsilon_1} E_0 r\cos\theta \,,$$

$$V(r \ge R) = -E_0 r\cos\theta + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + 2\varepsilon_1} \frac{R^3}{r^2} E_0 \cos\theta \,,$$

where θ is the angle the position vector r makes with the direction of the external electric field and all the other symbols have their usual meaning.

Using the above information,

(a) Find out the electric field inside a spherical cavity of radius R which is hollowed out from an infinite dielectric media of permittivity ε. The whole system is placed in an electric field which is uniform far away from the sphere and is given by $\vec{E} = E_0 \hat{z}$. Comment on the magnitude and direction of the electric field with respect to the external field.

[**3 marks**]

(b) Find out the electric field outside the spherical cavity but inside the dielectric media. [**3 marks**]

(c) Plot the magnitude of electric field along the z-axis. [**2 marks**]

(d) Sketch the electric field lines. [**2 marks**]

Assume isotropic, linear and homogeneous dielectrics.

Question 3

(**A**) The rate of a particular chemical reaction $A + B \to C$ is proportional to the concentrations of the reactants A and B. Given that $C(t = 0) = 0$, and

$dC(t)/dt = \alpha\,[A(0) - C(t)]\,[B(0) - C(t)]$, where α is a constant.

(a) Find $C(t)$ for $A(0) \neq B(0)$. [**4 marks**]

(b) Find $C(t)$ for $A(0) = B(0)$. [**3 marks**]

(**B**) Given that m is an integer, and $f(z) = z^m$, calculate the contour integral of $f(z)$ over a unit circle, with origin at $z = 0$. [**3 marks**]

Question 4

(A) A particle of mass m is constrained to move on a curve in the vertical plane defined by the parametric equation: $x = l(2\phi + \sin 2\phi)$; $y = l(1 - \cos 2\phi)$. There is the usual constant gravitational force acting in the vertical y direction.

(a) Calculate the Hamiltonian of the system. Is the Hamiltonian conserved? Is the energy of the system conserved? For each case give proper justification to your answer. [**3 marks**]

(b) Calculate the action integral for the system. [**4 marks**]

(**B**) Three equal mass points (mass 10 g) are located at (a, 0, 0); (0, a, 2a); and (0, 2a, a). Obtain the principal moments of inertia of the system. Take a = 2 cm. [**3 marks**]

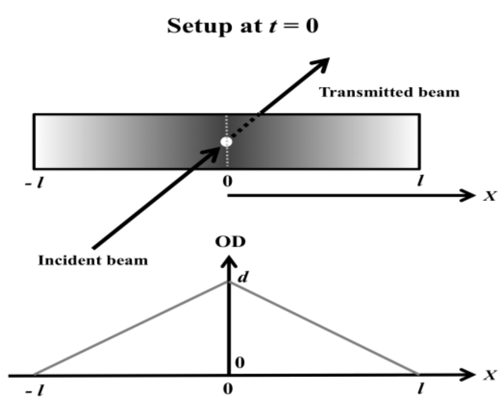
Question 5

(**A**) A digital stopwatch can read at a precision of 1/10 of a second. However, the display of the watch is damaged and the tens' place of second is not readable (the display looks like: 00:00:X0.0). Where "X" represents the tens place of a second which is not readable. What is the effective measurement precision of this digital stopwatch? Explain your answer briefly.

[**2 marks**]

(B) Random measurement uncertainties are inevitably introduced in any measurement and are propagated to the processed data. The time period (T) of a pendulum is measured in two different ways. In one experiment the total time for 50 oscillations (T_{50}) is measured and the time period is calculated as $T = (T_{50} / 50)$. In another experiment, time for each complete oscillation (T_1) is measured 50 times and the time period is calculated by taking mean, *i.e.* $T = (<T_1>_{50})$. Compare the propagated uncertainties in these two cases and thus conclude which between the two, statistically, gives more accurate value for the time period? [**3 marks**]

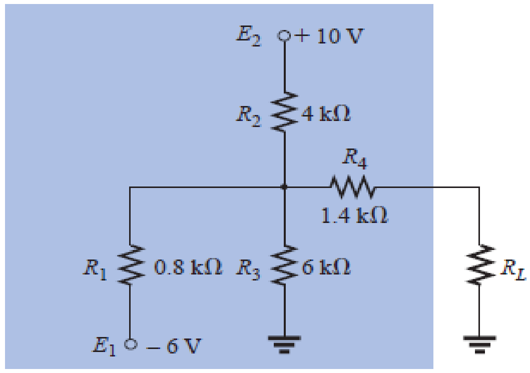
(C) When a light beam of intensity I_0 passes through a neutral density (ND) filter, the intensity of the transmitted light (I_t) gets reduced by a factor $10^{-\eta}$ i.e. $I_t = I_0\, 10^{-\eta}$, where η is the optical density of the filter. In an experiment a rectangular ND filter (length = $2l$) is used, where η changes linearly from a maximum value of d at the center to 0 at both ends ($\pm\, l$) along its length (see figure below). <u>A laser beam is passed through the middle of this ND filter</u>. Now, if the ND filter starts performing simple harmonic motion along the length with time period T and amplitude l. Derive the transmitted intensity of the laser beam as a function of time. What is the time period of oscillation in the transmitted intensity? Does it oscillate in a simple harmonic manner? What is the minimum time that it needs to be averaged over to calculate the time averaged transmitted intensity? [**5 marks**]



Question 6

(A) Find the Thevenin equivalent circuit (across R_L) for the following network:

[5 marks]



(B) Draw the circuit diagram for negative feedback amplifiers of following specifications using an ideal Op-Amp (IC-741). Each circuit must contain three (and only three) 10 kΩ resistors.

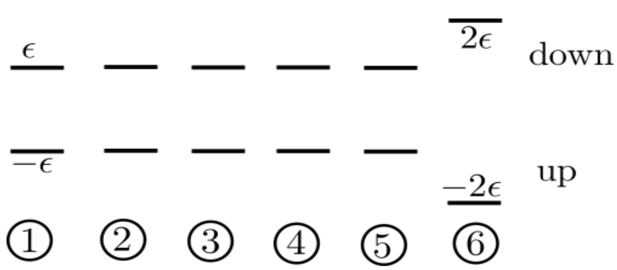
[5 marks]

(a) $A_{v(CL)}$ = -2 and R_I = 10 kΩ.
(b) $A_{v(CL)}$ = -2 and R_I = 5 kΩ.
(c) $A_{v(CL)}$ = -0.5 and R_I = 10 kΩ.
(d) $A_{v(CL)}$ = +3
(e) $A_{v(CL)}$ = +3 and R_F = 10 kΩ.

Here, $A_{v(CL)}$ is the closed loop gain. R_I is the input resistor and R_F is the feedback resistor.

Question 7

Consider a system of six distinguishable, non-interacting spins. Each spin can only occupy two states: `up' and `down'. For the first five spins, the energy levels are $-\varepsilon$ for an up spin and $+\varepsilon$ for down spin. However, the sixth spin has twice the magnetic moment and, therefore, it's energy levels are -2ε and $+2\varepsilon$. If the total energy is -3ε, calculate (a) the entropy and (b) the average number of up spins. **[7marks + 3 marks]**



Useful formulae (In spherical coordinates):

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$