	Indian Institute of Technology Kanpur Department of Mathematics and Statistics **Written Test for Ph.D. admissions in Mathematics**					
Maximum Marks : 90	Date : December 3, 2018		Time : 90 Minutes			
Name of the Candidate						
Roll Number		Category (Tick One)	GEN	OBC	SC/ST/PwD	

INSTRUCTIONS

(1) There are three sections; the first section has TRUE/FALSE questions, the second section is fill in the blanks and the third section has multiple choice questions.
- In the first section, every correct answer will be awarded 2 marks and a wrong answer will be awarded NEGATIVE 1 (-1) marks.
- In the second section, every correct answer will be awarded 2 marks and a wrong answer will be awarded 0 marks.
- The third section has one or two correct answers. In this section
  - each question has four choices.
  - if a wrong answer is selected in a question then that entire question will be awarded 0 marks.
  - the candidate gets full credit of 4 marks, only if he/she selects all the correct answers and no wrong answers; 2 mark will be awarded for an answer to a question if it is partially correct and a wrong answer is not selected.

(2) These question-cum-answer-sheets must be returned to the invigilator before leaving the examination hall.
(3) **Please enter your answers on this page in the space given below.**

True/False Questions

Q. No.	Correct Option
1	
2	
3	
4	
5	
6	
7	
8	
9	

Fill In The Blanks Questions

Q. No.	Answer	Q. No.	Answer
1		7	
2		8	
3		9	
4		10	
5		11	
6		12	

Multiple Choice Questions

Q. No.	Correct Option(s)	Q. No.	Correct Option(s)	Q. No.	Correct Option(s)	Q. No.	Correct Option(s)
1		4		7		10	
2		5		8		11	
3		6		9		12	

Notations

I. We denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, the set of natural numbers, real numbers and complex numbers, respectively.

True/False [18 marks]

(1) Consider $\mathbb{R}$ with the co-countable topology τ, which consists of the empty set and all subsets A such that $\mathbb{R} \setminus A$ is countable. Then $(\mathbb{R}, \tau)$ is Hausdorff.

(2) Let $T : H \to H$ be a bounded linear operator on a real Hilbert space H. Suppose $\langle Tx, x\rangle = 0$, for all $x \in H$. Then $T = 0$.

(3) Consider the function space

$$C^{\frac{1}{2}}([0,1]) := \{f : [0,1] \to \mathbb{R} : \exists K > 0 \text{ such that } |f(x) - f(y)| \leq K|x-y|^{\frac{1}{2}}, \forall x, y \in [0,1]\}.$$

Then both $x \mapsto \ln(1+x)$ and $x \mapsto x^2$ are elements of $C^{\frac{1}{2}}([0,1])$.

(4) Consider the second order ordinary differential equation (ODE)

$$2x^3 y''(x) + (\cos 2x - 1)y'(x) + 2xy(x) \ = \ 0.$$

The number of independent Frobenius series solution is exactly 1.

(5) Let $Y(x)$ be a bounded solution of the ordinary differential equation (ODE)

$$(1-x^2)y'' - 2xy' + 6y = 0.$$

If $Y(1) = 2$, then $\displaystyle\int_{-1}^{1} Y(x)dx = \int_{-1}^{1} xY(x)dx$.

(6) Every nontrivial solution of $y''(x) + (1 + \sin^2 x + \cos^4 x)y(x) \ = \ 0$ has only finite number of zeros.

(7) There exists a non abelian group of order 18.

(8) Let A be a 4×4 real matrix such that $A^3 = I$. Then 1 necessarily is an eigen value of A.

(9) Let R be a ring such that $x^2 = 2x$ for all $x \in R$. Then $4x = 0$ for all $x \in R$.

Fill in the blanks [24 marks]

(1) If C is the circle $\{z \in \mathbb{C} : |z+2| = 3\}$ oriented anti-clockwise, then value of the integral

$$\int_C \frac{dz}{z^3(z+4)}$$

is __________ .

(2) If

$$a_n := \begin{cases} \frac{1}{3^n}, n \text{ is odd}, \\ \frac{1}{5^n}, n \text{ is even}, \end{cases}$$

then the radius of convergence of the power series $\sum_n a_n z^n$ is __________ .

(3) Consider the sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined by

$$f_n(x) := n^2 x e^{-n^2x^2} - (n-1)^2 x e^{-(n-1)^2x^2}, x \in [0,1].$$

Then the value of the integral $\int_0^1 \sum_{n=1}^{\infty} f_n(x)\, dx$ is __________ .

(4) Let A be the set of all holomorphic functions f from $\mathbb{C} \setminus \{0\}$ onto the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$. Then the cardinality of the set A is __________ .

(5) Given that $e^x f(y)$ is an integrating factor of

$$y' + \sin y + x \cos y + x = 0.$$

If $f(0) = 1$, then $f(y) =$ __________ .

(6) If a continuous function $y(x)$ satisfies

$$y(x) + \int_0^x (2 + x - t) y(t)\, dt = 1 + 2x,$$

then $y(x) =$ __________ .

(7) Let $f(x)$ be a continuous function which has exactly one zero in the interval $(2, 6)$. The minimum number of iteration of the bisection method so that the zero of $f(x)$ can be determined with an accuracy of 2^{-20} is __________ .

(8) For a function f, the following values and divided differences are given:

$$f(3) = 19, f[1,3] = 9, f[0,1,3] = 3.$$

Then $f(0) =$ __________ .

(9) Let G be a group of order 25 in which every element has order either 1 or 5. Then the number of subgroups of order 5 in G is __________ .

(10) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix}$. Then the product of all the eigen values of A is __________ .

(11) Number of elements which are not invertible in the ring of integers $\{0, 1, 2, \ldots, 19\}$ modulo 20 is __________ .

(12) Number of 7-Sylow subgroups of S_7 is __________ .

Questions with one or two correct choices [48 marks]

(1) Let $\mathbb{N}$, the set of natural numbers, be endowed with the metric

$$d(n,m) := \left| \frac{1}{n} - \frac{1}{m} \right|, n, m \in \mathbb{N}.$$

Then

a. All functions $f : (\mathbb{N}, d) \to (\mathbb{R}, |\cdot|)$ are continuous, where $|\cdot|$ denotes the usual Euclidean metric on $\mathbb{R}$.
b. The space $(\mathbb{N}, d)$ is complete.
c. The space $(\mathbb{N}, d)$ is compact.
d. The space $(\mathbb{N}, d)$ is connected.

(2) Let $\phi : (X, d_1) \to (Y, d_2)$ be a homeomorphism between two metric spaces. Then

a. $\phi(A)$ is bounded subset of Y, whenever A is a bounded subset of X.

b. $\phi(A^\circ) = (\phi(A))^\circ$, for $A \subseteq X$, where A° denotes the interior of A.

c. $\phi(\bar{A}) = \overline{\phi(A)}$, for $A \subseteq X$, where $\bar{A}$ denotes the closure of A.

d. $d_1(x, y) = d_2(\phi(x), \phi(y))$ for all $x, y \in X$.

(3) If $f : [0, \infty) \to [0, \infty)$ is uniformly continuous, then

a. f^2 is uniformly continuous on $[0, \infty)$.

b. $f \circ f$ is uniformly continuous on $[0, \infty)$.

c. $F(x) := \int_0^x f(t)\, dt, x \in [0, \infty)$ is uniformly continuous on $[0, \infty)$.

d. f maps Cauchy sequences (of non negative real numbers) to Cauchy sequences.

(4) Let ℓ^1 and ℓ^2 be the spaces of real sequences defined as follows.

$$\ell^1 := \{x = \{x_n\} : \sum_{n=1}^{\infty} |x_n| < \infty\},$$

$$\ell^2 := \{x = \{x_n\} : \sum_{n=1}^{\infty} x_n^2 < \infty\}.$$

Consider the metrics d_1 and d_2 on ℓ^1 and ℓ^2 respectively, as follows.

$$d_1(x, y) := \sum_{n=1}^{\infty} |x_n - y_n|$$

and

$$d_2(x, y) := \left(\sum_{n=1}^{\infty} (x_n - y_n)^2 \right)^{\frac{1}{2}}.$$

Then

a. Both ℓ^1 and ℓ^2 are separable.

b. ℓ^1 is separable and ℓ^2 is not separable.

c. Neither ℓ^1 nor ℓ^2 is separable.

d. ℓ^1 is not separable and ℓ^2 is separable.

(5) Consider the one dimensional wave equation

$$u_{tt} = u_{xx}, \quad 0 < x < \infty,\ t > 0,$$

$$u(0, t) = 0, \quad t \geq 0; \qquad u_t(x, 0) = 0, \quad 0 \leq x < \infty,$$

$$u(x, 0) = \begin{cases} \sin^2\left(\frac{\pi x}{2}\right), & 2 \leq x \leq 4 \\ 0, \quad 0 \leq x \leq 2, & x \geq 4. \end{cases}$$

Then,

a. $u(4, 1) = \frac{1}{2}$ **b.** $u(4, 1) = 1$ **c.** $u(\frac{9}{2}, 1) = \frac{1}{4}$ **d.** $u(\frac{9}{2}, 1) = \frac{1}{2}$

(6) Consider the ordinary differential equation (ODE)

$$y''' + ay'' + by' + cy = 0, \qquad x \in (-\infty, \infty),$$

where a, b and c are arbitrary constants. Then, which of the following (is) are NOT solution of this ODE

a. $x \sin x$ **b.** x^3 **c.** $x^2 e^x$ **d.** xe^x

(7) I is the approximate value of the integral $\Delta = \int_{-2}^{2} ||x+1|-|x-1||dx$ obtained using Trapezoidal rule with four equispaced subintervals. Then

a. $I = \Delta$ **b.** $\Delta = 6$ **c.** $I = 0$ **d.** $I \neq \Delta$

(8) Let $f(t)$, for $t \geq 0$, be a continuous function and it is of exponential order in t. Let $F(s)$ be the Laplace transform of f and F satisfies $s^2F''(s) - 6F(s) = 0$. If $f(2) = 8$, then

a. $f(1) = 4$ **b.** $f(1) = 2$ **c.** $f(4) = 16$ **d.** $f(4) = 32$

(9) Let S_7 denote the group of permutations of $\{1, 2, \ldots, 7\}$. Let $\sigma \in S_7$ be an element such that it has highest order among all the elements of S_7. Then the order of σ is

a. 7 **b.** 10 **c.** 12 **d.** 15

(10) Which of the following has a solution (a, b, c) such that a, b, c are integers?

a. $5x + 8y + 17z = 5$

b. $171x + 102y + 87z = 9$

c. $42x + 84y + 105z = 6$

d. $21x + 42y + 3z = 1$

(11) Let R_1 be the ring of continuous real valued functions on the interval $[0, 1]$ and R_2 be the ring of entire functions on $\mathbb{C}$ (an entire function is a function from $\mathbb{C}$ to $\mathbb{C}$ which is analytic every where). Then

a. R_1 is an integral domain and R_2 is not an integral domain.

b. R_2 is an integral domain and R_1 is not an integral domain.

c. Both R_1 and R_2 are integral domains.

d. Neither R_1 nor R_2 is an integral domain.

(12) Let R be a finite integral domain and $f\colon R \longrightarrow R$ be a non-zero ring homomorphism. Then

a. f need not be a surjective map.

b. f need not be an injective map.

c. f is necessarily an isomorphism.

d. f, in general, is neither an injective map nor a surjective map.