Grade Table (for checker use only)

Question	Points	Score
1	10	
2	5	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	15	
Total:	100	

Team Members:

- ....
- ....
- ....

INSTRUCTIONS:

- **Write your team name on top of each page.**
- If you have any queries, contact an invigilator. Any sort of interaction with another team can lead to a penalty or disqualification.
- Submit any electronic devices that you possess, to one of the invigilators. You may collect them after the event. Any team caught using any electronic device will be immediately disqualified.
- Enough space has been provided in the question paper. Use it wisely. However, if you need extra sheets, contact an invigilator.

1. (10 points) Two IITK alumni (Mr. Ashu and Mr. Apurv) bump into each other after over 20 years from their graduation.
   Ashu: "How have you been?"
   Apurv: "Great! I got married and I have three daughters now."
   Ashu: "Really? How old are they?"
   Apurv: "Well, the product of their ages is 72, and the sum of their ages is the same as the number on that building over there."
   Ashu: "Right, ok.. oh wait..hmm, I still don't know."
   Apurv: "Oh sorry, the oldest one just started to play the piano."
   Ashu: "Wonderful! My oldest is the same age!"
   What are the ages of Apurv's daughters?

   Sloution:
   The product of the ages of the three daughters being 72 give the following possible triplets.

Triplet	Sum
$(1, 1, 72)$	74
$(1, 2, 36)$	39
$(1, 3, 24)$	28
$(1, 4, 18)$	23
$(1, 6, 12)$	19
$(1, 8, 9)$	18
$(2, 2, 18)$	22
$(2, 3, 12)$	17
$(2, 4, 9)$	15
$(2, 6, 6)$	14
$(3, 3, 8)$	14
$(3, 4, 6)$	13

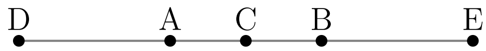
After knowing the sum of the ages, Ashu is still unable to figure out the correct triplet which means that the sum has occurred twice. So, 14 is the required sum.
Now, Apurv tells that there is a unique oldest daughter which means that $(3, 3, 8)$ is the required age triplet.

2. (5 points) 50 integers are chosen among the first 100 positive integers $(1, 2, ..., 100)$ in such a manner that no two of the chosen integers have a sum equal to 100. Show that there is at least one perfect square among the numbers chosen.

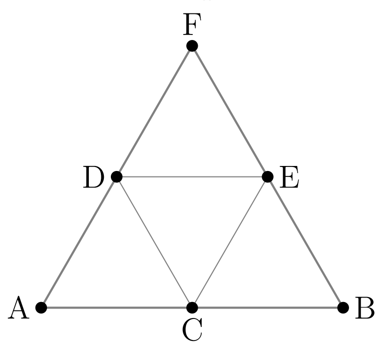
   Solution:
   Consider the sets $\{1, 99\}, \{2, 98\}, \{3, 97\}, ..., \{36, 64\}, ..., \{49, 51\}, \{50\}, \{100\}$. These are a total of 51 sets. Since, the sum of no two of the chosen numbers is 100. Therefore, from each set at most one number can be chosen.
   Also, since we have to select 50 numbers, there is exactly one set from which no number is chosen. In other words, one number is chosen either form $\{36, 64\}$ or $\{100\}$, which gives a perfect square.

3. (10 points) Points on a plane are coloured either *red* or *blue*. Prove that
   (a) there is a line segment whose mid point and end points are all of the same colour.
   (b) there is an equilateral triangle whose vertices are all of the same colour.

Solution: part (a):
Take any two points of the same colour, say A and B which are both *red* (WLOG).
Let their mid point be C.



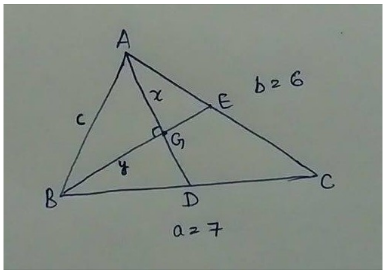
If C is *red*, we are done.
If C is *blue*, take two other points D and E on the line AB such that $DA = AB$ and $AB = BE$.
If D is *red*, DB is the required segment.
Similarly, if E is *red*, AE is the required segment.
Otherwise, if both D and E are *blue*, then DE is the required segment.
part (b):
Using first part, take one such line segment, say AB with mid point C. WLOG assume they are *red*.
Consider the equilateral triangles shown in the figure.



If any of the points D, E, F is *red*, we will get one of the *red* triangles ($\Delta DAC, \Delta ECB, \Delta FAB$).
If all of D, E, F are blue, we will get a *blue* triangle (ΔDEF).

4. (10 points) In a ΔABC, the median from A is perpendicular to the median from B. If $BC = 7$ and $AC = 6$, find AB.

Solution:



Let AD, BE be the medians from A, B resp. and let G be the centroid.
Let $AG = x$ and $BG = y$.
In ΔABG, by Pythagoras theorem,

$$c^2 = x^2 + y^2$$

$$Ar(\Delta ABG) = \frac{1}{3}Ar(\Delta ABC)$$

$$\implies \sqrt{\frac{(13+c)}{2}\frac{(13-c)}{2}\frac{(c+1)}{2}\frac{(c-1)}{2}} = 3 \times \frac{1}{2}xy$$

Also, since centroid divides the median in the ratio of 2 : 1, by Pythagoras theorem in ΔAEG,

$$x^2 + (\frac{y}{2})^2 = 3^2$$

Solving, we get $c = \sqrt{17}$.

5. (10 points) Find all positive real numbers x, y, z such that

$$2x - 2y + \frac{1}{z} = \frac{1}{2017}, \quad 2y - 2z + \frac{1}{x} = \frac{1}{2017}, \quad 2z - 2x + \frac{1}{y} = \frac{1}{2017}.$$

Solution:
Adding the three equations, we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{2017}.$$

We can also write the equations in the form

$$2xz - 2yz + 1 = \frac{z}{2017}, \quad 2yx - 2zx + 1 = \frac{x}{2017}, \quad 2zy - 2yx + 1 = \frac{y}{2017}.$$

Adding these, we also get

$$2017 \times 3 = x + y + z.$$

Therefore

$$\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)(x + y + z) = \frac{3}{2017} \times (2017 \times 3) = 9.$$

Using AM-GM inequality, we therefore obtain

$$9 = \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)(x + y + z) \geq 9 \times (xyz)^{\frac{1}{3}} \left(\frac{1}{xyz}\right)^{\frac{1}{3}} = 9.$$

Hence equality holds in AM-GM inequality and we conclude $x = y = z$. Thus

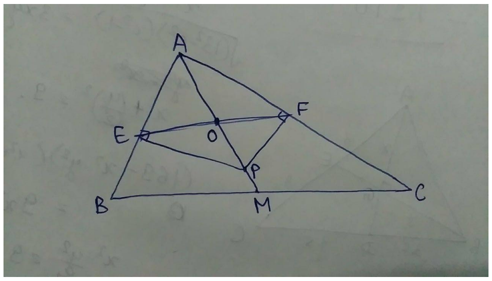
$$\frac{1}{x} = \frac{1}{2017}$$

which gives $x = 2017$. We conclude

$$x = 2017, \quad y = 2017, \quad z = 2017.$$

6. (10 points) In a ΔABC, M is the mid-point of BC, P is any point on AM and PE, PF are perpendiculars to AB, AC respectively. If EF is parallel to BC, find out the value of $\angle A$.

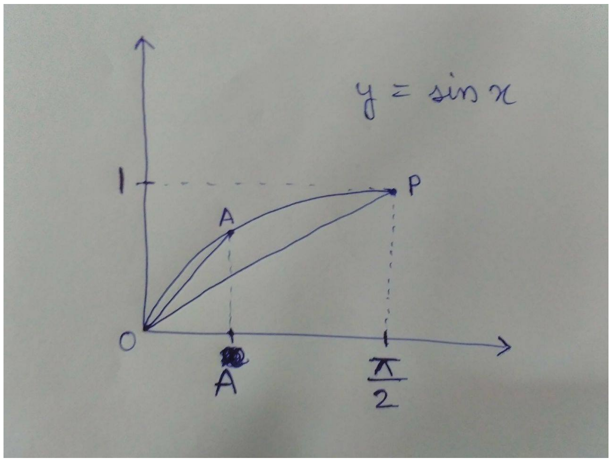
Solution:



Since, $\angle AEP + \angle AFP = 180°$, $AEPF$ is a cyclic quadrilateral.
Also, $EF \parallel BC \implies OE = OF$.
Since, diameter AP bisects AP.
Either $AP \perp EF$ which gives isosceles triangle and infinite possible values of $\angle A$.
Or EF is another diameter of the circle, which gives $\angle A = 90°$.

7. (10 points) In an acute angled triangle ABC, prove that $\sin A + \sin B + \sin C > 2$.

Solution:



Since angle A is less than $\pi/2$, it can be seen from the graph that

$$Slope(OA) > Slope(OP)$$

.

$$\therefore \frac{\sin A}{A} > \frac{2}{\pi}$$
$$\implies \sin A > \frac{2A}{\pi}$$

Adding results for A, B, C, we get

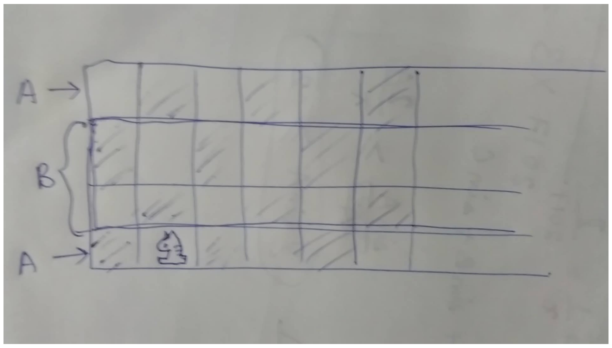
$$\sin A + \sin B + \sin C > 2\Big(\frac{A+B+C}{\pi}\Big) = 2$$

8. (10 points) Let $d(n)$ denote the number of positive divisors of the positive integer n. Determine those numbers n for which $d(n^3) = 5d(n)$.

Solution:
Let the number n have the prime factorization $n = p_1^{a_1} p_2^{a_2} ... p_k^{a_k}$, p_i's are primes and a_i's are positive integers.
Then, $n^3 = p_1^{3a_1} p_2^{3a_2} ... p_k^{3a_k}$.
For each i from 1 to k, the power of prime p_i in the factor can be from 0 to a_i.
$\therefore d(n) = (a_1 + 1)(a_2 + 1)...(a_k + 1)$.
and $d(n^3) = (3a_1 + 1)(3a_2 + 1)...(3a_k + 1)$.
Note that for each i from 1 to k,
$3a_i + 1 > 2(a_i + 1)$
$\implies d(n^3) > 2^k d(n)$
$\implies 5d(n) > 2^k d(n)$
$\implies k = 1$ or $k = 2$
$k = 1$ gives
$5(a + 1) = 3a + 1$
which is not possible for positive a.
$k = 2$ gives
$5(a_1 + 1)(a_2 + 1) = (3a_1 + 1)(3a_2 + 1)$
$\implies (2a_1 - 1)(2a_2 - 1) = 5$
$\implies a_1 = 1$ and $a_2 = 3$ (WLOG).
So, n is any number of the form $p_1 p_2^3$, where p_1, p_2 are any prime numbers.

9. (10 points) Is is possible for a knight to start on any square of a $4 \times n$ chessboard, visit every square exactly once and return back to its original square?

Solution:



Consider the two parts of the $4 \times n$ chessboard, A and B as shown in the figure. Note that from a square in A, a knight cannot go to another square in A. It has to go to B. Since the number of B squares are equal to the number of A squares.
To complete knight tour, it has to go from a square in B to a square in A.
That is, a knight has to move from A to B and the from B to A and continue like this.
Also note that a knight always moves from a light square to a dark square or vice versa in a move.
Therefore, it can only cover either all the dark or all the light squares of each part.

10. (15 points) Two players A and B take turns removing chips from a pile that initially contains n chips. The first player A cannot remove all the n chips at the beginning. A player, on his turn, must remove at least one chip. However he cannot remove more chips than those his opponent removed on his previous turn. The player who removes the last chip is the winner. Assuming both players play optimally, give the condition on n such that B always wins.

    Solution: Claim : B wins iff $n = 2^m$ for some positive integer m.
    ($\implies$ part) :
    A can't remove more than or equal to $n/2$ chips. Otherwise, B would remove the rest chips and win immediately.
    Also if A removes an odd number of chips, B can remove one chip. In all the following turns, each player is forced to remove exactly one chip. By parity argument, B removes the last chip and wins.
    So A removes an even number of chips and which is less than $n/2$. Suppose the number of chips remaining is $2^k q$, where q is an odd number ≥ 3. B removes 2^k chips such that number of chips remaining is $2^k(q-1)$. This way, in each step the power of 2 in the number of chips remaining before A's turn is greater than that before B's turn.
    Also note that no. of chips remaining at A's turn is greater than the no. of chips removed by B in the previous turn. $[2^k(q-1) > 2^k$ as $q \geq 3]$. Continuing with this strategy, it will be A's turn whenever the number of chips is a power of 2. Eventually, B will remove last chip and win.
    ($\impliedby$ part) :
    If the number of chips to begin with is not a power of 2. Let 2^k be the largest power of 2 smaller than n.
    The winning startegy for A is to remove that many chips so that 2^k chips are remaining. B can't remove all chips at once. If both players play optimally, the game proceeds as the previous case but the roles of A and B are reversed. So, A wins in this case.