Grade Table (for checker use only)

Question	Points	Score
1	8	
2	10	
3	14	
4	14	
5	18	
6	18	
7	22	
8	26	
9	30	
10	30	
Total:	190	

Team Members (Name and Roll no):

- ....
- ....
- ....

<u>INSTRUCTIONS:</u>

- **Write your team name on top of each page.**
- If you have any queries, contact an invigilator. Any sort of interaction with another team can lead to a penalty or disqualification.
- Submit any electronic devices that you possess, to one of the invigilators. You may collect them after the event. Any team caught using any electronic device will be immediately disqualified.
- Enough space has been provided in the question paper. Use it wisely. However, if you need extra sheets, contact an invigilator.

1. (8 points) Define the function $f(x, y, z)$ by

$$f(x, y, z) = x^{y^z} - x^{z^y} + y^{z^x} - y^{x^z} + z^{x^y}$$

Evaluate $f(1, 2, 3) + f(1, 3, 2) + f(2, 3, 1) + f(2, 1, 3) + f(3, 1, 2) + f(3, 2, 1)$
Solution :
Take $g(x, y, z) = f(x, y, z) - z^{x^y}$. By symmetry, $g(1, 2, 3) + g(1, 3, 2) + g(2, 1, 3) + g(2, 3, 1) + g(3, 1, 2) + g(3, 2, 1) = 0$. So the desired sum is $x^{y^z} + x^{z^y} + y^{z^x} + y^{x^z} + z^{x^y} + z^{y^x} = 24$

2. (10 points) Start with n equally spaced dots $P_1, P_2, ...P_n$ on a straight line , with a distance 1 between consecutive dots. Using P_1P_2 as a base side, draw a regular pentagon in the plane. Similarly, draw $n-2$ additional regular pentagons on the base side P_1P_3, P_1P_4, ..., P_1P_n, all pentagons lying on the same side of the the line P_1P_n. Find the total number of dots.
   Solution :
   Ans (4n-3).
   No. of points on the line=n
   No. of points on the pentagon(excluding the base points)=3
   Total no. of pentagons=(n-1)
   Total no. of points on the pentagon(excluding the base points)=3(n-1)
   Total no. of points=No. of points on the base line + No. of points on the remaining parts=n+3n-3=4n-3

3. (14 points) An urn contains 1729 balls of different colors. Randomly select a pair, repaint the first to match the second, and replace the pair in the urn. What is the expected time until the balls are all the same color?
   Solution
   Ans=1728^2
   If the color classes have sizes $k_1, k_2, ..., k_m$
   For a particular k
   Define $\phi(k) = (\frac{k(k-1)}{2} \sum_{j=1}^{k} j)$
   We first show that $\phi(k+1) + \phi(k-1) - 2\phi(k) = (n-1)/(n-k)$ except when k=n; the $\frac{k(k-1)}{2}$ contributes 1, the term j=k contributes $(j-1)/(n-j) = (k-1)/(n-k)$ and the other summations $j < k$ contribute nothing.
   Then we say that the expected change in $\phi(k)$ on a give color class is $\frac{k(n-k)}{(n(n-1))}$ times $(\phi(k+1) + \phi(k-1) - 2\phi(k))$, since with probability $frack(n-k)(n(n-1))$ the class goes to size k+1 and with
   the same probability it goes to size k-1. This expected change comes out to k/n.
   Summing over the color classes (and remembering the minus sign),
   the expected change in the "cost from here" on one step is -1, except when we're already monochromatic,
   where the exception k=n kicks in. One can rewrite the contribution from as $(n-1)\sum_{j=1}^{k}(k-j)/(n-j)$ which incorporates both the k(k-1)/2 and the previous sum over j. Class size=k Adding over all the classes. We get $(n-1)^2$.Substitute n=1729
   Therefore, $Ans = 1728^2$

4. (14 points) In trapezoid ABCD, with sides AB and CD parallel, $\angle DAB = 6°$ and $\angle ABC = 42°$. Point X on side AB is such that $\angle AXD = 78°$ and $\angle CXB = 66°$. If AB and CD are 1 unit apart, Find AD + DX- BC - CX

   Solution :
   Ans=8 units
   Dropping a perpendicular (of length 1) from D to AX, and similarly from C to BX, we see that:
   $AD = \text{cosec}(6)$
   $DX = \text{cosec}(78)$
   $BC = \text{cosec}(42)$
   $CX = \text{cosec}(66)$
   Notice that, for $x = 6, 78, 42, 66, and 30$, $\sin(5x) = 0.5$. We now express $\sin(5x)$ in terms of $\sin(x)$.
   De Moivre's Theorem states that for any real number x and any integer n,
   $\cos(nx) + i\sin(nx) = (\cos(x) + i\sin(x))^n$
   Setting n = 5, expanding the right-hand side using the binomial theorem, and equating imaginary parts, we obtain
   $\sin(5x) = \sin^5(x) - 10\sin^3(x)\cos^2(x) + 5\sin(x)\cos^4(x) = \sin^5(x) - 10\sin^3(x)(1 - \sin^2(x)) + sin(x)(1 - \sin^2(x))^2$
   Since, $\sin^2(x) + \cos^2(x) = 1 = 16\sin^5(x)20\sin^3(x) + 5\sin(x)$ This result can also be obtained by means of trigonometric identities.
   Setting $s = \sin(x)$, it follows that the five distinct real numbers,
   $\sin(6), \sin(78), \sin(42), \sin(66), and \sin(30) = \frac{1}{2}(1)$
   are roots of the equation
   $16s^5 - 20s^3 + 5s = \frac{1}{2}$, or, equivalently, of $32s^5 - 40s^3 + 10s - 1 = 0$ (2)
   By the Fundamental Theorem of Algebra, (2) has exactly five roots, up to multiplicity, and hence these must be precisely the distinct roots identified in (1). Since $s = \frac{1}{2}$ is a root of (2), the equation factorizes:
   $(2s - 1)(16s^4 + 8s^3 - 16s^2 8s + 1) = 0$
   yielding the quartic equation whose roots are $\sin(6), \sin(78), \sin(42), and \sin(66)$. As s = 0 is not a root of this quadratic equation, we may divide by s^4, and, setting $t = \frac{1}{s}$, obtain
   $t^4 - 8t^3 16t^2 + 8t + 16 = 0$
   an equation whose roots are $\text{cosec}(6), \text{cosec}(78), -\text{cosec}(42), -\text{cosec}(66)$. By Viète's formulas, the sum of the roots of this equation is 8.
   Thus, AD + DX (BC + CX) = 8 inches.

   P.S.:Viète's formulas is just the fancy name for the rule about the relation between the co-efficients and the roots of a polynomial:-)

5. (18 points) An immortal flea jumps on whole points of the number line, beginning with 0. The length of the first jump is 3, the second 5, the third 9, and so on. The length of k^{th} jump is equal to $2^k + 1$. The flea decides whether to jump left or right on its own. Is it possible that sooner or later the flea will have been on every natural point, perhaps having visited some of the points more than once?
   Solutions :
   It's enough to prove that being at the point x at its k-th move, the flea can make some jumps and after that to reach $x \pm 1$. Indeed, let it jumps $m+1$ times to the right and the last $m+2$-th jump be to the left. Thus it would be at the point:

$$x + (2^k + 1) + (2^{k+1} + 1) + \cdots + (2^{k+m} + 1) - (2^{k+m+1} + 1) = x - 2^k + m.$$

   Now, choosing $m = 2^k \pm 1$, we prove the claim.

6. (18 points) 100 integers are arranged in a circle. Each number is greater than the sum of the two subsequent numbers (in a clockwise order). Determine the maximal possible number of positive numbers in such circle.
   Solutions:
   Label the numbers $a_1, a_2, \ldots, a_{100}$. Then $a_i > a_{i+1} + a_{i+2}$ for all i, if we consider the indices $\pmod{100}$. Now assume that there are at least 50 positive numbers on the circle. Then two positive numbers will follow a non-positive one (which contradicts the given condition) unless every other number is non-positive. Hence there are at most 50 positive numbers on the circle, and according to the discussion above, we may assume wlog that they are $a_1, a_3, \ldots, a_{99}$.

   But, adding some inequalities gives

   $$a_{100} > a_1 + a_2 > a_1 + a_3 + a_4 > a_1 + a_3 + a_5 + a_6 > \cdots > \sum_{2 \nmid i} a_i + a_{100} \iff 0 > \sum_{2 \nmid i} a_i$$

   so at least one of $a_1, a_2, \ldots, a_{99}$ is negative as well, implying that at most 49 of the numbers on the circle are positive. This is the number we are looking for: For the 98 first a_i, let

   $$a_i = \begin{cases} 1 & \text{if } i \text{ is odd,} \\ -998 - i & \text{if } i \text{ is even,} \end{cases}$$

   and set $a_{99} = -700, a_{100} = -500$. This does indeed work, and we are done - the answer is $\boxed{49}$.

   Just a comment: This can probably be generalized for any even number of numbers on a circle satisfying the same conditions. A similar construction will do, but the negative numbers will have to have a little larger absolute value.[/quote]

7. (22 points) Prove that $\dfrac{[(2+\sqrt{3})^{2n-1}]-1}{2}$ is a perfect square for all $n \in N$. Where $[x]$ is the greatest integer function
Solutions:
Just note that, $(2+\sqrt{3})^{2n-1}+(2-\sqrt{3})^{n} \in \mathbb{N}$. Also, $(2-\sqrt{3})^{2n-1} < 1$. Thus,

$$[(2+\sqrt{3})^{2n-1}] = (2+\sqrt{3})^{2n-1} + (2-\sqrt{3})^{2n-1} - 1$$

Now, suppose, $\frac{(2+\sqrt{3})^{2n-1}+(2-\sqrt{3})^{2n-1}-2}{2} = x_n^2$ We have, $x_1 = 1, x_2 = 5$ and it is easy to check that, $x_{n+1} = 4x_n - x_{n-1}$. Thus, x_n is a sequence of integers. So we are done

8. (26 points) Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$.
   Solution :
   For $a \geq b$ then it suffices to check for $b \leq a \leq 1$. If $a \geq 2$ then it's obvious that $3a \geq 2b + 2$. For $a < b$, we have $1 + \frac{b!}{a!} \mid b! = \frac{b!}{a!} \cdot a!$ implies $1 + \frac{b!}{a!} \mid a!$.

   On the other hand, since there are $b - a + 1$ consecutive numbers in the product $\frac{b!}{a!}$ so for any prime p such that $1 \leq p \leq b - a$ then $p \mid \frac{b!}{a!}$. Therefore, $\gcd\left((b-a)!, \frac{b!}{a!} + 1\right) = 1$.

   If $b - a + 1$ is a composite number then $\gcd\left((b-a+1)!, \frac{b!}{a!} + 1\right) = 1$. This follows $1 + \frac{b!}{a!} \mid \frac{a!}{(b-a+1)!}$, which means

$$\frac{a!}{b!} = \underbrace{(a+1)\cdots b}_{b-a \text{ numbers}} < \underbrace{(b-a+2)\cdots a}_{2a-b-1 \text{ numbers}} = \frac{a!}{(b-a+1)!}.$$

   Since $b > a$ so each factor in LHS is greater than the respective factor in RHS, so the only way for the above inequality to be true is $b - a < 2a - b - 1$ or $3a \geq 2b + 2$.

   If $b - a + 1$ is a prime. If $b - a + 1 \neq a$ then there always exist a positive integer k such that $a \leq k(b - a + 1) \leq b$. This means $b - a + 1 \mid \frac{b!}{a!}$. We do the similar thing like the above case. If $b - a + 1 = a$ is a prime. From $\gcd\left((b-a)!, \frac{b!}{a!} + 1\right) = 1$ we follow $b - a < 2a - b$ or $3a \geq 2b + 1$. Since $2a - 1 = b$ so $a \leq 1$. This case can be easily checked.

   Thus, $3a \geq 2b + 2$ as desired.

9. (30 points) Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.

Solutions :

Let $P(x, y)$ be the assertion $f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$

$P(0, 0) \Rightarrow f(f(0)) = 0$

$P(0, f(0)) \Rightarrow 2f(0) = f(0)^2 \Rightarrow f(0) = 0$ or 2

case 1.$f(0) = 2 \Rightarrow f(2) = 0$

$P(x, 1) \Rightarrow f(x + f(x + 1)) = x + f(x + 1)$

$P(0, f(x + 1) + x) \Rightarrow f(x + 1) + x + 2 = f(x + 1) + x + 2(f(x + 1) + x)$

$\Rightarrow f(x) = 2 - x \forall \in \mathbb{R}$

case 2.$f(0) = 0$

$P(x, 0) \Rightarrow f(x + f(x)) = x + f(x)$

$P(x, 1) \Rightarrow f(x + f(x + 1)) = x + 1 + f(x)$

$P(1, f(x + 1) + x) \Rightarrow f(1 + f(1 + x + f(x + 1))) + f(x + f(x + 1)) = 1 + f(x + 1 + f(x + 1)) + f(x + 1) + x$

$\Rightarrow f(f(x) + x + 1) = f(x) + x + 1$

$P(x, -1) \Rightarrow f(x + f(x - 1)) + f(-x) = x + f(x - 1) - f(x)$

$\Rightarrow -f(x) = f(-x)$

$P(x, -x) \Rightarrow f(x) + f(-x^2) = x - xf(x)$

$P(-x, x) \Rightarrow f(-x) + f(-x^2) = -x + x(-x)$

$\Rightarrow f(x) - f(-x) = 2x - x(f(x) + f(-x))$

$\Rightarrow f(x) = x \forall x \in \mathbb{R}$

hence, $f(x) = 2 - x$ and $f(x) = x$ are solutions.

10. (30 points) Prove that

$$|\cos(x)|+|\cos(y)|+|\cos(z)|+|\cos(y+z)|+|\cos(z+x)|+|\cos(x+y)|+3|\cos(x+y+z)| \geq 3$$

for all real x, y, and z.
Solution :
Claim : for all x,y $\in R$, we have $|\cos(x)| + |\cos(y)| + |cos(x + y)| >= 1$. Proof :

$$|\cos(x)|+|\cos(y)|+|\cos(x+y)| \geq |\cos(x)||\sin(y)|+|\cos(y)||\sin(x)|+|\cos(x+y)| \geq |\sin(x+y)|+|\cos(x+$$

Using this claim, we have following chain of inequalities,

$$|\cos(x)| + |\cos(y + z)| + |cos(x + y + z)| >= 1$$

$$|\cos(y)| + |\cos(x + z)| + |cos(x + y + z)| >= 1$$

$$|\cos(z)| + |\cos(x + y)| + |cos(x + y + z)| >= 1$$

Adding them gives the desired result .